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Algebraic Structures on Two-Dimensional Vector Space Over Any Basic Field

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Abstract

In the paper we utilize a new approach to the classification problem of finite-dimensional algebras. We give a complete classifications of associative and diassociative algebra structures on two-dimensional vector space over any basic field.

Keywords: associative algebra; diassociative algebra; classification.

1 Introduction

In 1993, Loday introduced the notion of Leibniz algebra [10], which is a generalization of Lie algebra, where the skew-symmetric of the bracket is dropped and the Jacobi identity is changed by the Leibniz identity. Loday noted that the link between Lie algebras and associative algebras can be extended to an "analogous" link between Leibniz algebra and so-called dialgebra which is a generalization of associative algebra possessing two products. Namely, it was shown that if one has a dialgebra $(D, \dashv, \vdash)$ over a finite-dimensional vector space V, with two bilinear binary operations with certain compatibility axioms then introducing a binary operation,

$$[x, y] := x \dashv y - y \vdash x,$$

we get an algebra structure on V called Leibniz algebra. It also been shown that the universal enveloping algebra of a Leibniz algebra has the structure of a dialgebra.

The main motivation of Loday to introduce these classes of algebras was the search of an "obstruction" to the periodicity in algebraic K-theory. Since then the study of different properties, relations and classification of Loday's algebras became an active research area. Dozens of papers have been published [1, 2, 5], also see [6] and the references therein). But most of the results have been related to algebras over the field of complex numbers. Recently, a result on classification of all algebra structures on two-dimensional vector space over any basic field was published [9]. In this paper we use the result of [9] to classify all associative and diassociative algebra structures on two-dimensional vector space over any basic field. This technique was implemented earlier in the series of papers [3, 4, 8] and others. However, there was a condition about the ground field, which was cleaned up in [9].

The organization of the paper is as follows. In Section 1 we give a short review on the classification problem of finite-dimensional algebras. Section 2 contains the main idea and results which will be using in the research. Section 3 is devoted to a complete classification of two-dimensional associative algebras over any basic fields. In Section 4 we compute the automorphism groups of associative algebras found in Section 3. Section 5 gives a complete classification of two-dimensional associative dialgebras over any basic field and Section 6 summarizes the results obtained.

2 Preliminaries

Definition 2.1. *A vector space* $\mathbb{V}$ *over a field* $\mathbb{F}$ *equipped with a function* $\cdot : \mathbb{V} \otimes \mathbb{V} \to \mathbb{V}\,((\mathrm{x}, \mathrm{y}) \mapsto \mathrm{x} \cdot \mathrm{y})$ *such that,*

$$(\alpha \mathrm{x} + \beta \mathrm{y}) \cdot \mathrm{z} = \alpha(\mathrm{x} \cdot \mathrm{z}) + \beta(\mathrm{y} \cdot \mathrm{z}), \qquad \mathrm{z} \cdot (\alpha \mathrm{x} + \beta \mathrm{y}) = \alpha(\mathrm{z} \cdot \mathrm{x}) + \beta(\mathrm{z} \cdot \mathrm{y}),$$

whenever $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathbb{V}$ *and* $\alpha, \beta \in \mathbb{F}$*, is said to be an algebra* $\mathbb{A} = (\mathbb{V}, \cdot)$.

Definition 2.2. *Two algebras* $\mathbb{A} = (\mathbb{V}, \cdot_{\mathbb{A}})$ *and* $\mathbb{B} = (\mathbb{V}, \cdot_{\mathbb{B}})$ *are called isomorphic if there is an invertible linear map* $f : \mathbb{V} \to \mathbb{V}$ *such that,*

$$f(\mathrm{x} \cdot_{\mathbb{A}} \mathrm{y}) = f(\mathrm{x}) \cdot_{\mathbb{B}} f(\mathrm{y}),$$

where $\mathrm{x}, \mathrm{y} \in \mathbb{A}$.

Definition 2.3. *An invertible linear map* $f : \mathbb{V} \to \mathbb{V}$ *is said to be an automorphism if,*

$$f(\mathrm{x} \cdot \mathrm{y}) = f(\mathrm{x}) \cdot f(\mathrm{y}),$$

whenever $\mathrm{x}, \mathrm{y} \in \mathbb{A} = (\mathbb{V}, \cdot)$.

The set of all automorphisms of an algebra $\mathbb{A}$ forms a group with respect to the composition operation and it is denoted by $Aut(\mathbb{A})$.

Let $\mathbb{A} = (\mathbb{V}, \cdot)$ be an n-dimensional algebra over $\mathbb{F}$ and $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n)$ be its basis. Then the bilinear map $\cdot : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ is represented by a $n \times n^2$ matrix (called the matrix of structure constant, shortly MSC);

$$A = \begin{pmatrix} a_{11}^1 & a_{12}^1 & \cdots & a_{1n}^1 & a_{21}^1 & a_{22}^1 & \cdots & a_{2n}^1 & \cdots & a_{n1}^1 & a_{n2}^1 & \cdots & a_{nn}^1 \\ a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 & a_{21}^2 & a_{22}^2 & \cdots & a_{2n}^2 & \cdots & a_{n1}^2 & a_{n2}^2 & \cdots & a_{nn}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{11}^n & a_{12}^n & \cdots & a_{1n}^n & a_{21}^n & a_{22}^n & \cdots & a_{2n}^n & \cdots & a_{n1}^n & a_{n2}^n & \cdots & a_{nn}^n \end{pmatrix},$$

as follows:

$$\mathrm{e}_i \cdot \mathrm{e}_j = \sum_{k=1}^{n} a_{ij}^k \mathrm{e}_k, \qquad \text{where } \ i, j = 1, 2, \ldots, n.$$

Therefore, the product on $\mathbb{A}$ with respect to the basis $\mathbf{e}$ is written as below,

$$\mathrm{x} \cdot \mathrm{y} = \mathbf{e}A(x \otimes y), \tag{1}$$

for any $\mathrm{x} = \mathbf{e}x$, $\mathrm{y} = \mathbf{e}y$, where $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ are column coordinate vectors of x and y, respectively, $x \otimes y$ is the tensor (Kronecker) product of x and y. Now and onward for the product "$\mathrm{x} \cdot \mathrm{y}$" on $\mathbb{A}$ we use the juxtaposition "xy".

Further, we assume that the basis $\mathbf{e}$ is fixed and we do not make a difference between the algebra $\mathbb{A}$ and its MSC A.

An automorphism $\mathrm{g} : \mathbb{A} \to \mathbb{A}$ as an invertible linear map is represented on the basis $\mathbf{e}$ by an invertible $n \times n$ matrix $g \in M(n; \mathbb{F})$ and $\mathrm{g}(\mathrm{x}) = \mathrm{g}(\mathbf{e}x) = \mathbf{e}gx$. Due to

$$\mathrm{g}(\mathrm{xy}) = \mathrm{g}(\mathbf{e}A(x \otimes y)) = \mathbf{e}g(A(x \otimes y)) = \mathbf{e}(gA)(x \otimes y),$$

and

$$\mathrm{g}(\mathrm{x})\mathrm{g}(\mathrm{y}) = (\mathbf{e}gx)(\mathbf{e}gy) = \mathbf{e}A(gx \otimes gy) = \mathbf{e}Ag^{\otimes 2}(x \otimes y).$$

The condition $\mathrm{g}(\mathrm{xy}) = \mathrm{g}(\mathrm{x})\mathrm{g}(\mathrm{y})$ is written in terms of A and g as follows,

$$gA = Ag^{\otimes 2}. \tag{2}$$

Note that, in this term Definition 2.2 can also be rewritten as,

$$gA = Bg^{\otimes 2} \Longleftrightarrow A = g^{-1}Bg^{\otimes 2}. \tag{3}$$

Definition 2.4. *An algebra* $(\mathbb{A}, \cdot)$ *is said to be associative if for all* $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathbb{A}$ *the following axiom is satisfied,*

$$(\mathrm{xy})\mathrm{z} = \mathrm{x}(\mathrm{yz}). \tag{4}$$

Write,

$$\mathrm{xy} = \mathbf{e}A(x \otimes y), \qquad \text{and} \qquad \mathrm{yz} = \mathbf{e}A(y \otimes z),$$
$$(\mathrm{xy})\mathrm{z} = \mathbf{e}A(A(x \otimes y) \otimes z), \quad \text{and} \quad \mathrm{x}(\mathrm{yz}) = \mathbf{e}A(x \otimes A(y \otimes z)).$$

Then,

$$\mathbf{e}A\Big(A(x\otimes y)\otimes z\Big)=\mathbf{e}A\Big(x\otimes A(y\otimes z)\Big),$$

i.e., an algebra $\mathbb{A}$ with MSC A is associative if and only if,

$$A(A\otimes I)=A(I\otimes A), \tag{5}$$

where I is $n\times n$ identity matrix.

Definition 2.5. *A dialgebra $\mathbb{D}=(\mathbb{V},\dashv,\vdash)$ is said to be an associative dialgebra if the following axioms are satisfied for all $\mathrm{x},\mathrm{y},\mathrm{z}\in\mathbb{D}$;*

$$\begin{aligned}(\mathrm{x}\dashv\mathrm{y})\dashv\mathrm{z}&=\mathrm{x}\dashv(\mathrm{y}\dashv\mathrm{z}),\\ \mathrm{x}\dashv(\mathrm{y}\dashv\mathrm{z})&=\mathrm{x}\dashv(\mathrm{y}\vdash\mathrm{z}),\\ (\mathrm{x}\vdash\mathrm{y})\dashv\mathrm{z}&=\mathrm{x}\vdash(\mathrm{y}\dashv\mathrm{z}),\\ (\mathrm{x}\dashv\mathrm{y})\vdash\mathrm{z}&=(\mathrm{x}\vdash\mathrm{y})\vdash\mathrm{z},\\ (\mathrm{x}\vdash\mathrm{y})\vdash\mathrm{z}&=\mathrm{x}\vdash(\mathrm{y}\vdash\mathrm{z}).\end{aligned} \tag{6}$$

Definition 2.6. *Let $\mathbb{D}_1=(\mathbb{V},\dashv,\vdash)$ and $\mathbb{D}_2=(\mathbb{V},\dashv',\vdash')$ be diassociative algebras. A linear function $f:\mathbb{V}\longrightarrow\mathbb{V}$ is said to be a homomorphism, if*

$$f(\mathrm{x}\dashv\mathrm{y})=f(\mathrm{x})\dashv' f(\mathrm{y}),\quad \textit{and}\quad f(\mathrm{x}\vdash\mathrm{y})=f(\mathrm{x})\vdash' f(\mathrm{y}),\quad \textit{for all}\quad \mathrm{x},\mathrm{y}\in\mathbb{V}.$$

Definition 2.7. *Dialgebras $\mathbb{D}_1$ and $\mathbb{D}_2$ are called isomorphic if there is an invertible homomorphism $f:\mathbb{D}_1\longrightarrow\mathbb{D}_2$.*

Lemma 2.1. *A dialgebra $\mathbb{D}$ with MSC $D:=\{A,B\}$ is diassociative if and only if the matrices A and B satisfy the following equations:*

$$\begin{aligned}A(A\otimes I)-A(I\otimes A)&=0,\\ A(I\otimes A)-A(I\otimes B)&=0,\\ A(B\otimes I)-B(I\otimes A)&=0,\\ B(A\otimes I)-B(B\otimes I)&=0,\\ B(B\otimes I)-B(B\otimes I)&=0.\end{aligned} \tag{7}$$

Proof. Let,

$$\mathrm{x}\dashv\mathrm{y}=\mathbf{e}A(x\otimes y)\text{ and }\mathrm{x}\vdash\mathrm{y}=\mathbf{e}B(x\otimes y),$$

for any $\mathrm{x}=\mathbf{e}x$, $\mathrm{y}=\mathbf{e}y$.

Then,

$$\begin{aligned}(\mathrm{x}\dashv\mathrm{y})\dashv\mathrm{z}&=\mathbf{e}A\Big(A(x\otimes y)\otimes z\Big),\\ \mathrm{x}\dashv(\mathrm{y}\dashv\mathrm{z})&=\mathbf{e}A\Big(x\otimes A(y\otimes z)\Big),\\ \mathrm{x}\dashv(\mathrm{y}\vdash\mathrm{z})&=\mathbf{e}A\Big(x\otimes B(y\otimes z)\Big),\\ (\mathrm{x}\vdash\mathrm{y})\dashv\mathrm{z}&=\mathbf{e}A\Big(B(x\otimes y)\otimes z\Big),\\ \mathrm{x}\vdash(\mathrm{y}\dashv\mathrm{z})&=\mathbf{e}B\Big(x\otimes(A(y\otimes z))\Big),\\ (\mathrm{x}\dashv\mathrm{y})\vdash\mathrm{z}&=\mathbf{e}B\Big(A(x\otimes y)\otimes z\Big),\end{aligned}$$

$$
\begin{aligned}
(\mathrm{x} \vdash \mathrm{y}) \vdash \mathrm{z} &= \mathbf{e}B\Big(B(x \otimes y) \otimes z\Big),\\
\mathrm{x} \vdash (\mathrm{y} \vdash \mathrm{z}) &= \mathbf{e}B\Big(x \otimes (B(y \otimes z))\Big).
\end{aligned}
$$

Therefore, the diassociative algebra axioms (6) in terms of the structure constants can be given by the identities:

$$
\begin{aligned}
A\Big(A(x \otimes y) \otimes z\Big) &= A\Big(x \otimes A(y \otimes z)\Big),\\
A\Big(x \otimes A(y \otimes z)\Big) &= A\Big(x \otimes B(y \otimes z)\Big),\\
A\Big(B(x \otimes y) \otimes z\Big) &= B\Big(x \otimes (A(y \otimes z))\Big),\\
B\Big(A(x \otimes y) \otimes z\Big) &= B\Big(B(x \otimes y) \otimes z\Big),\\
B\Big(B(x \otimes y) \otimes z\Big) &= B\Big(x \otimes (B(y \otimes z))\Big).
\end{aligned} \tag{8}
$$

By the property $(M \otimes N)(P \otimes S) = MP \otimes NS$ of the tensor product these equations can be rewritten as (7). □

The result that we are going to use in the paper from [9] was given as follows.

Theorem 2.1. *Any non-trivial 2-dimensional algebra over a field $\mathbb{F}$ ($Char(\mathbb{F}) \neq 2, 3$) is isomorphic to only one of the following listed, by their matrices of structure constants, such algebras:*

- $A_1(\mathrm{c}) = \begin{pmatrix} a_1 & a_2 & 1+a_2 & a_4 \\ b_1 & -a_1 & 1-a_1 & -a_2 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_2, a_4, b_1) \in \mathbb{F}^4$.
- $A_2(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & b_2 & 1-a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$ *and* $a_4 \neq 0$.
- $A_3(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & b_2 & 1-a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2 a_4 \\ 0 & b_2 & 1-a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ *and* $a \neq 0$.
- $A_4(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & b_2 & 1 & -1 \end{pmatrix}$, *where* $\mathrm{c} = (b_1, b_2) \in \mathbb{F}^2$.
- $A_5(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 1 & 2a_1 - 1 & 1-a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = a_1 \in \mathbb{F}$.
- $A_6(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & 1-a_1 & -a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$ *and* $a_4 \neq 0$.
- $A_7(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & 1-a_1 & -a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2 a_4 \\ 0 & 1-a_1 & -a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$.
- $A_8(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathrm{c} = b_1 \in \mathbb{F}$.
- $A_9 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$.

- $A_{10}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1^{'}(a) & 0 & 0 & -1 \end{pmatrix}$,
  where $\mathrm{c} = b_1 \in \mathbb{F}$, *the polynomial* $(b_1t^3 - 3t - 1)(b_1t^2 + b_1t + 1)(b_1^2t^3 + 6b_1t^2 + 3b_1t + b_1 - 2)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $b_1^{'}(t) = \dfrac{(b_1^2t^3 + 6b_1t^2 + 3b_1t + b_1 - 2)^2}{(b_1t^2 + b_1t + 1)^3}$.

- $A_{11}(\mathrm{c}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ b_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3b_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, *where the polynomial* $b_1 - t^3$ *has no root in* $\mathbb{F}$, $a, \mathrm{c} = b_1 \in \mathbb{F}$ *and* $a, b_1 \neq 0$.

- $A_{12}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2b_1 & 0 & 0 & -1 \end{pmatrix}$, *where* $a, \mathrm{c} = b_1 \in \mathbb{F}$ *and* $a \neq 0$.

- $A_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 2.2. *Any non-trivial 2-dimensional algebra over a field* $\mathbb{F}$ $(Char(\mathbb{F}) = 2)$ *is isomorphic to only one of the following listed by their matrices of structure constants, such algebras:*

- $A_{1,2}(\mathrm{c}) = \begin{pmatrix} a_1 & a_2 & 1 + a_2 & a_4 \\ b_1 & a_1 & 1 + a_1 & a_2 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_2, a_4, b_1) \in \mathbb{F}^4$.

- $A_{2,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & b_2 & 1 + a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$ *and* $a_4 \neq 0$.

- $A_{2,2}(a_1, 0, 1) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 1 & 1 & 1 + a_1 & 0 \end{pmatrix}$, *where* $a_1 \in \mathbb{F}$.

- $A_{3,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & b_2 & 1 + a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2a_4 \\ 0 & b_2 & 1 + a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ *and* $a \neq 0$.

- $A_{4,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 1 & 1 & 0 \\ b_1 & b_2 & 1 + a_1 & 1 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 1 & 1 & 0 \\ b_1 + (1 + b_2)a + a^2 & b_2 & 1 + a_1 & 1 \end{pmatrix}$,
  where $\mathrm{c} = (a_1, b_1, b_2) \in \mathbb{F}^3$.

- $A_{5,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & 1 + a_1 & a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$ *and* $a_4 \neq 0$.

- $A_{5,2}(1, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$.

- $A_{6,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & 1 + a_1 & a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2a_4 \\ 0 & 1 + a_1 & a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$.

- $A_{7,2}(\mathrm{c}) = \begin{pmatrix} a_1 & 1 & 1 & 0 \\ b_1 & 1 + a_1 & a_1 & 1 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 1 & 1 & 0 \\ b_1 + aa_1 + a + a^2 & 1 + a_1 & a_1 & 1 \end{pmatrix}$,
  where $\mathrm{c} = (a_1, b_1) \in \mathbb{F}^2$ *and* $a \in \mathbb{F}$.

- $A_{8,2}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1^{'}(a) & 0 & 0 & 1 \end{pmatrix}$, *where the polynomial* $(b_1t^3 + t + 1)(b_1t^2 + b_1t + 1)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $b_1^{'}(t) = \dfrac{(b_1^2t^3 + b_1t + b_1)^2}{(b_1t^2 + b_1t + 1)^3}$.

- $A_{9,2}(\mathrm{c}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ b_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3 b_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, *where* $a, \mathrm{c} = b_1 \in \mathbb{F}$ *and* $a \neq 0$, *the polynomial* $b_1 + t^3$ *has no root in* $\mathbb{F}$.
- $A_{10,2}(\mathrm{c}) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ b_1 & 1 & 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 & 0 \\ b_1 + a + a^2 & 1 & 1 & 1 \end{pmatrix}$, *where* $a, \mathrm{c} = b_1 \in \mathbb{F}$.
- $A_{11,2}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(b_1 + a^2) & 0 & 0 & 1 \end{pmatrix}$, *where* $a, b \in \mathbb{F}$ *and* $b \neq 0$.
- $A_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 2.3. *Any non-trivial* 2*-dimensional algebra over a field* $\mathbb{F}$ $(Char(\mathbb{F}) = 3)$ *is isomorphic to only one of the following, listed by their matrices of structure constants, such algebras:*

- $A_{1,3}(\mathrm{c}) = \begin{pmatrix} a_1 & a_2 & a_2 + 1 & a_4 \\ b_1 & -a_1 & 1 - a_1 & -a_2 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_2, a_4, b_1) \in \mathbb{F}^4$.
- $A_{2,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & b_2 & 1 - a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$, *and* $a_4 \neq 0$.
- $A_{3,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & b_2 & 1 - a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2 a_4 \\ 0 & b_2 & 1 - a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4, b_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ *and* $a \neq 0$.
- $A_{4,3}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & b_2 & 1 & -1 \end{pmatrix}$, *where* $\mathrm{c} = (b_1, b_2) \in \mathbb{F}^2$.
- $A_{5,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 1 & 2a_1 - 1 & 1 - a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = a_1 \in \mathbb{F}$.
- $A_{6,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 1 & 1 - a_1 & -a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$ *and* $a_4 \neq 0$.
- $A_{7,3}(\mathrm{c}) = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & 1 - a_1 & -a_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} a_1 & 0 & 0 & a^2 a_4 \\ 0 & 1 - a_1 & -a_1 & 0 \end{pmatrix}$, *where* $\mathrm{c} = (a_1, a_4) \in \mathbb{F}^2$, $a \in \mathbb{F}$ *and* $a \neq 0$.
- $A_{8,3}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 1 & 0 & -1 \end{pmatrix}$, *where* $\mathrm{c} = b_1 \in \mathbb{F}$.
- $A_{9,3}(b_1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ b_1'(a) & 0 & 0 & -1 \end{pmatrix}$, *where the polynomial* $(b_1 - t^3)(b_1 t^2 + b_1 t + 1)(b_1^2 t^3 + b_1 - 2)$ *has no root in* $\mathbb{F}$, $a \in \mathbb{F}$ *and* $b_1'(t) = \dfrac{(b_1^2 t^3 + b_1 - 2)^2}{(b_1 t^2 + b_1 t + 1)^3}$.
- $A_{10,3}(\mathrm{c}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ b_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3 b_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$,
  where the polynomial $b_1 - t^3$ *has no root*, $a, \mathrm{c} = b_1 \in \mathbb{F}$ *and* $a, b_1 \neq 0$.
- $A_{11,3}(\mathrm{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ b_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2 b_1 & 0 & 0 & -1 \end{pmatrix}$, *where* $a, \mathrm{c} = b_1 \in \mathbb{F}$, $a \neq 0$.
- $A_{12,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}$.
- $A_{13,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

3 Classification of Two-Dimensional Associative Algebras

In the paper we make use the results of [9] above as Theorem 1, Theorem 2 and Theorem 3 for the characteristic of the basic field to be not 2, 3 (Theorem 1), to be 2 (Theorem 2) and to be 3 (Theorem 3). This and the sections followed are devoted to the classification of associative and diassociative algebra structures on two-dimensional vector space over any basic field relying on the theorems above.

Let $\mathbb{A}$ be a two-dimensional associative algebra and

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix},$$

be its MSC on a basis $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2)$. Write the axiom (5) in terms of the elements of A as follows:

$$\begin{cases} \beta_1(\alpha_2 - \alpha_3) = 0, \\ \alpha_2\beta_2 - \alpha_4\beta_1 = 0, \\ (\alpha_1 - \beta_3)\alpha_2 - \alpha_3(\alpha_1 - \beta_2) = 0, \\ (\alpha_1 - \beta_2)\alpha_4 - \alpha_2(\alpha_2 - \beta_4) = 0, \\ \alpha_3\beta_3 - \alpha_4\beta_1 = 0, \\ \alpha_4(\beta_2 - \beta_3) = 0, \\ (\alpha_1 - \beta_3)\alpha_4 - \alpha_3(\alpha_3 - \beta_4) = 0, \\ \alpha_4(\alpha_2 - \alpha_3) = 0, \\ \beta_1(\beta_2 - \beta_3) = 0, \\ (\alpha_2 - \beta_4)\beta_1 - \beta_2(\alpha_1 - \beta_2) = 0, \\ (\alpha_3 - \beta_4)\beta_1 - \beta_3(\alpha_1 - \beta_3) = 0, \\ (\alpha_3 - \beta_4)\beta_2 - \beta_3(\alpha_2 - \beta_4) = 0. \end{cases} \tag{9}$$

Theorems 1, 2 and 3 from [9] are applied as follows: substitute the structure constants of the list of representatives in the theorems into the system of equations (9) taking the structure constants to be variables. The solutions to the system give the structure constants of associative algebras.

3.1 The characteristic is not 2 and 3

Theorem 3.1. *Any non-trivial 2-dimensional associative algebra over a field $\mathbb{F}$, with the characteristic not 2 and 3, is isomorphic to one of the following algebras presented by their matrices of structure constants:*

1. $As_{13}^1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$
2. $As_3^2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$
3. $As_3^3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$

4. $As_3^4 := \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$.

5. $As_3^5(\alpha_4) := \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \simeq \begin{pmatrix} \frac{1}{2} & 0 & 0 & a^2\alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$, *where* $\alpha_4 \in \mathbb{F}$, $a \in \mathbb{F}$ *and* $a \neq 0$.

Proof. In this case we will be dealing with Theorem 3.1.

It is easy to see that A_{13} is associative.

For algebras $A_{12} - A_4$ the system of equations (9) is inconsistent.

Consider,

$$A_3(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix},$$

where $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ and $a \neq 0$.

Then, we get the system of equations,

$$\begin{cases} (\alpha_1 - \beta_2)\alpha_4 = 0, \\ \alpha_4(\alpha_1 + \beta_2 - 1) = 0, \\ \alpha_4(2\alpha_1 - 1) = 0, \\ \beta_2(\alpha_1 - \beta_2) = 0, \\ 2\alpha_1^2 - 3\alpha_1 + 1 = 0, \\ \alpha_4(\alpha_1 + \beta_2 - 1) = 0. \end{cases}$$

But, $2\alpha_1^2 - 3\alpha_1 + 1 = 0$ if and only if $\alpha_1 = 1$ or $\alpha_1 = \frac{1}{2}$. This produces the cases:

Case 1: $\alpha_1 = 1:$

$$\begin{cases} \alpha_4(\beta_2 - 1) = 0, \\ \alpha_4\beta_2 = 0, \\ \alpha_4 = 0, \\ \beta_2(\beta_2 - 1) = 0, \\ \alpha_4\beta_2 = 0, \end{cases} \quad \text{we get} \quad \beta_2(\beta_2 - 1) = 0.$$

Case 11: If $\beta_2 = 0$, then we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case 12: If $\beta_2 = 1$, then we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Case 2: If $\alpha_1 = \dfrac{1}{2}$, one has

$$\begin{cases} \alpha_4(2\beta_2 - 1) = 0, \\ 2\beta_2^2 - \beta_2 = 0. \end{cases}$$

Case 21: $\alpha_4 = 0$ implies $2\beta_2^2 - \beta_2 = 0$. Therefore, we have;

Case 211, where $\beta_2 = 0$ and

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Case 212: If $\beta_2 = \dfrac{1}{2}$, then

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Case 22: $a_4 \neq 0$. This implies $\beta_2 = \dfrac{1}{2}$ and we obtain

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \simeq \begin{pmatrix} \frac{1}{2} & 0 & 0 & a^2\alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

where $\alpha_4 \in \mathbb{F}$, $a \in \mathbb{F}$ and $a \neq 0$.

Note that if $\alpha_4 \neq 0$ and $\mathbb{F}$ is perfect (particularly, algebraically closed), then $\alpha_4 = 1$.

For algebras A_1 and A_2 the system of equations (9) also is inconsistent. □

3.2 The characteristic is two

Theorem 3.2. *Any non-trivial 2-dimensional associative algebra over a field $\mathbb{F}$, with the characteristic 2, is isomorphic to one of the following algebras presented by their matrices of structure constants:*

1. $As^1_{12,2} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.
2. $As^2_{11,2}(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1 + a^2) & 0 & 0 & 1 \end{pmatrix}$, *where $a, b \in \mathbb{F}$ and $b \neq 0$.*
3. $As^3_{6,2} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.
4. $As^4_{4,2}(\beta_1) := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 + a + a^2 & 0 & 0 & 1 \end{pmatrix}$, *where $a, \beta_1 \in \mathbb{F}$.*

5. $As^5_{3,2} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

6. $As^6_{3,2} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Proof. We verify the algebras given in Theorem 3.2 to be associative.

All the equations of the system (9) for algebras

$$A_{11,2}(c) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1 + a^2) & 0 & 0 & 1 \end{pmatrix}, \text{ where } a, b \in \mathbb{F},\ b \neq 0,$$

and

$$A_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

in Theorem 2.2 become identities. Therefore, $A_{11,2}$ and $A_{12,2}$ are associative algebras.

The algebra,

$$A_{4,2} := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \text{ is associative.}$$

It is easy to see that the algebra,

$$A_{6,2} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ also is associative.}$$

□

3.3 The characteristic is three

Theorem 3.3. *Any non-trivial* 2*-dimensional associative algebra over a field* $\mathbb{F}$*, with the characteristic* 3*, is isomorphic to one of the following algebras presented by their matrices of structure constants:*

1. $As^1_{13,3} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

2. $As^2_{3,3} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

3. $As^3_{3,3} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

4. $As^4_{3,3} := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$.

5. $As^5_{3,3}(\alpha_4) := \begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix} \cong \begin{pmatrix} 2 & 0 & 0 & a^2\alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix}$*, where* $\alpha_4 \in \mathbb{F}$*,* $a \in \mathbb{F}$ *and* $a \neq 0$.

Proof. In this case the associative algebras come out from the list of Theorem 3.3 as follows;

It is immediate to get that the algebra $A_{13,3}$ is associative. In these case all the equations of the system (9) turn into identities, whereas for algebras $A_{12,3} - A_{4,3}$ the corresponding systems of equations are inconsistent.

Let us consider,

$$A_{3,3}(\mathrm{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix} \cong \begin{pmatrix} \alpha_1 & 0 & 0 & a^2\alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix},$$

where $\mathrm{c} = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$, $a \in \mathbb{F}$ and $a \neq 0$.

The system of equations (9) is equivalent to

$$\begin{cases} (\alpha_1 - \beta_2)\alpha_4 = 0, \\ \alpha_4(\alpha_1 + \beta_2 - 1) = 0, \\ \alpha_4(2\alpha_1 - 1) = 0, \\ \beta_2(\alpha_1 - \beta_2) = 0, \\ 2\alpha_1^2 - 3\alpha_1 + 1 = 0, \\ \alpha_4(\alpha_1 + \beta_2 - 1) = 0. \end{cases} \tag{10}$$

The fifth equation of (10) is $2\alpha_1^2 - 3\alpha_1 + 1 = 0$, i.e., $\alpha_1 = 1$ or $\alpha_1 = 2$.

Case 1: $\alpha_1 = 1$ Then (10) is equivalent to $\beta_2^2 - \beta_2 = 0$. Therefore, we have two subcases:

Case 11: Let $\beta_2 = 0$. Then, we get

$$A_{3,3} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is associative.}$$

Case 12: Let $\beta_2 = 1$. Then, one obtains that

$$A_{3,3} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ is associative.}$$

Case 2: If $\alpha_1 = 2$ then (10) is equivalent to $\beta_2^2 - \beta_2 = 0$. Considering two subcases for $\beta_2 = 0$ (which implies $\alpha_4 = 0$) and $\beta_2 = 2$ we obtain the following two associative algebras:

$$A_{3,3} := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

and

$$A_{3,3} := \begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix} \cong \begin{pmatrix} 2 & 0 & 0 & a^2\alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix},$$

where $\alpha_4 \in \mathbb{F}$, $a \in \mathbb{F}$ and $a \neq 0$.

There are no associative algebras generated from the classes $A_{2,3}$ and $A_{1,3}$ of Theorem 3.3. □

4 The Automorphism Groups

In this section we describe the automorphism groups of algebras from Theorems 3.1, 3.2 and 3.3. It is sure that such automorphism groups can be obtained easily. But the lists of associative algebras in the theorems are over any basic field and we do it here for the paper to be self-contained. We need the automorphism groups to verify whether some of two-dimensional diassociative algebras found in Section 5 isomorphic or not.

Let $g = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ with $xt \neq yz$. The equation (2) is equivalent to

$$\begin{cases} \alpha_1 x^2 + \Big((\alpha_2+\alpha_3)z + \alpha_1\Big)x + \alpha_4 z^2 - \beta_1 y = 0, \\ \Big(\alpha_1 y + \alpha_2(t-1)\Big)x + (\alpha_3 z - \beta_2)y + \alpha_4 tz = 0, \\ \Big(\alpha_1 y + \alpha_3(t-1)\Big)x + (\alpha_2 z - \beta_3)y + \alpha_4 tz = 0, \\ \alpha_1 y^2 + \Big((\alpha_2+\alpha_3)t - \beta_4\Big)y + \alpha_4(t^2 - x) = 0, \\ \beta_4 z^2 + \Big((\beta_2+\beta_3)x - \alpha_1\Big)z + \beta_1(x^2 - t) = 0, \\ \Big(\beta_4 z + \beta_2(x-1)\Big)t + (\beta_3 y - \alpha_2)z + \beta_1 xy = 0, \\ \Big(\beta_4 z + \beta_3(x-1)\Big)t + (\beta_2 y - \alpha_3)z + \beta_1 xy = 0, \\ \beta_4 t^2 + \Big((\beta_2+\beta_3)y - \beta_4\Big)t + \beta_1 y^2 - \alpha_4 z = 0. \end{cases} \tag{11}$$

4.1 The characteristic of $\mathbb{F}$ is not 2 and 3

This section deals with the automorphism groups of associative algebras from Theorem 3.1.

Lemma 4.1. *The automorphism groups of 2-dimensional associative algebras over a field $\mathbb{F}$, $(Char(\mathbb{F}) \neq 2, 3)$ are given as follows:*

1. $Aut(As_{13}^1) = \left\{ \begin{pmatrix} x & 0 \\ z & x^2 \end{pmatrix} \mid x, z \in \mathbb{F},\ x \neq 0 \right\}.$
2. $Aut(As_3^2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{F},\ t \neq 0 \right\}.$
3. $Aut(As_3^3) = \left\{ \begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix} \mid t, z \in \mathbb{F},\ t \neq 0 \right\}.$
4. $Aut(As_3^4) = \left\{ \begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix} \mid z, t \in \mathbb{F},\ t \neq 0 \right\}.$
5. $Aut(As_3^5(0)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{F}, t \neq 0 \right\}.$
6. $Aut(As_3^5(\alpha_4)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}.$

Proof. For As^1_{13} the system (11) is equivalent to $\left\{ \begin{array}{r} y=0, \\ x^2-t=0. \end{array} \right.$ Therefore,

$$Aut(As^1_{13}) = Aut\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right) = \left\{ \left(\begin{array}{cc} x & 0 \\ z & x^2 \end{array} \right) \mid x \in \mathbb{F}, x \neq 0 \right\}.$$

Consider As^2_3. Then as the system (11) we get

$$\left\{ \begin{array}{r} x(x-1)=0, \\ y=0, \\ z=0. \end{array} \right.$$

Hence,

$$Aut(As^2_3) = Aut\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \left\{ \left(\begin{array}{cc} 1 & 0 \\ 0 & t \end{array} \right) \mid t \in \mathbb{F}, t \neq 0 \right\}.$$

Consider As^3_3. Then substituting the elements of MSC of As^3_3 into (11) we obtain

$$\left\{ \begin{array}{r} x-1=0, \\ y=0, \end{array} \right.$$

which implies

$$Aut(As^3_3) = Aut\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right) = \left\{ \left(\begin{array}{cc} 1 & 0 \\ z & t \end{array} \right) \mid t \in \mathbb{F}, t \neq 0 \right\}.$$

For $As^4_3 := \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$ also we get,

$$\left\{ \begin{array}{r} x-1=0, \\ y=0, \end{array} \right.$$

therefore,

$$Aut(As^4_3) = Aut\left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}\right) = \left\{ \left(\begin{array}{cc} 1 & 0 \\ z & t \end{array} \right) \mid t \neq 0 \right\}.$$

Let us now consider $As^5_3 := \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$. Then, the system of equations (11) becomes

$$\left\{ \begin{array}{r} x-x^2-2\alpha_4 z^2=0, \\ (x-1)y+2\alpha_4 zt=0, \\ 2x\alpha_4-y^2-2\alpha_4 t^2=0, \\ z-2xz=0, \\ (x-1)t+zy=0, \\ \alpha_4 z-ty=0. \end{array} \right.$$

The solution to the system is

$$\begin{cases} \{x=1, \quad y=0, \quad z=0, \quad t\neq 0\} & \text{if} \quad \alpha_4=0,\\ \{x=1, \quad y=0, \quad z=0, \quad t=\pm 1\} & \text{if} \quad \alpha_4\neq 0. \end{cases}$$

i.e.,

$$Aut(As_3^5(0)) = Aut\left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & 0\\ 0 & t \end{pmatrix} \mid t\neq 0\right\},$$

$$Aut(As_3^5(\alpha_4)) = Aut\left(\begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4\\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 & 0\\ 0 & \pm 1 \end{pmatrix}\right\}, \quad \text{where } \alpha_4\neq 0.$$

□

4.2 The characteristic of $\mathbb{F}$ is two

This section is devoted to the description of the automorphism groups of associative algebras from Theorem 3.2.

Lemma 4.2. *The automorphism groups of 2-dimensional associative algebras over a field $\mathbb{F}$, $(Char(\mathbb{F})=2)$ are given as follows:*

1. $Aut(As^1_{12,2}) = \left\{\begin{pmatrix} x & 0\\ z & x^2 \end{pmatrix} \mid x\neq 0, z\in\mathbb{F}\right\},$
2. $Aut(As^2_{11,2}) = \left\{\begin{pmatrix} x & 0\\ \beta_1(x-1) & 1 \end{pmatrix} \mid x\neq 0\in\mathbb{F}\right\},$
3. $Aut(As^3_{6,2}) = \left\{\begin{pmatrix} 1 & 0\\ z & t \end{pmatrix} \mid t\neq 0, z\in\mathbb{F}\right\},$
4. $Aut(As^4_{4,2}) = \left\{\begin{pmatrix} 1 & 0\\ z & 1 \end{pmatrix} \mid z\in\{0,1\}\right\},$
5. $Aut(As^5_{3,2}) = \left\{\begin{pmatrix} 1 & 0\\ 0 & t \end{pmatrix} \mid t\neq 0\in\mathbb{F}\right\},$
6. $Aut(As^6_{3,2}) = \left\{\begin{pmatrix} 1 & 0\\ z & t \end{pmatrix} \mid z,t\in\mathbb{F} \text{ and } t\neq 0\right\}.$

Proof. Consider $As^1_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}$. From (11) we get $y=0$ and $t=x^2$. Therefore,

$$Aut(As^1_{12,2}) = \left\{\begin{pmatrix} x & 0\\ z & x^2 \end{pmatrix} \mid \text{where } x\neq 0, z\in\mathbb{F}\right\}.$$

Let us take $As^2_{11,2}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1+a^2) & 0 & 0 & 1 \end{pmatrix}$, where $a, b \in \mathbb{F}$ and $b \neq 0$. Then we get

$$\begin{cases} t = 1, \\ y = 0, \\ z = \beta_1(x-1), \end{cases}$$

and

$$Aut(As^2_{11,2}) = \left\{ \begin{pmatrix} x & 0 \\ \beta_1(x-1) & 1 \end{pmatrix} \mid \text{where } x \neq 0 \in \mathbb{F} \right\}.$$

Consider $As^3_{6,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then (11) is equivalent to

$$\begin{cases} x - 1 = 0, \\ y = 0, \\ z = 0, \\ t - 1 = 0, \end{cases}$$

and we get

$$Aut(As^3_{6,2}) = \left\{ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Consider $As^4_{4,2}(\beta_1) := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 + a + a^2 & 0 & 0 & 1 \end{pmatrix}$. Then from the system of equations (11), we obtain

$$\begin{cases} \beta_1 y + x^2 + x = 0, \\ (t+y+1)x + yz = 0, \\ y(y+1) = 0, \\ (x^2+t)\beta_1 + z^2 + z = 0, \\ \beta_1 xy + tz + z = 0, \\ \beta_1 y^2 + t^2 + t = 0. \end{cases}$$

We get

$$Aut(As^4_{4,2}(\beta_1)) = \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \mid z \in \{0, 1\} \right\}.$$

Consider $As^5_{3,2} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then, (11) becomes

$$\begin{cases} x + 1 = 0, \\ y = 0, \\ z = 0. \end{cases}$$

Hence,

$$Aut(As^5_{3,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \neq 0 \in \mathbb{F} \right\}.$$

Consider $As^6_{3,2} := \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$. Then,

$$\left\{\begin{array}{r} y = 0,\\ xz + z = 0,\\ tx + t = 0.\end{array}\right.$$

Therefore,

$$Aut(As^6_{3,2}) = \left\{\left(\begin{array}{cc} 1 & 0\\ z & t\end{array}\right) \mid z \in \mathbb{F} \text{ and } t \neq 0 \in \mathbb{F}\right\}.$$

□

4.3 The characteristic of $\mathbb{F}$ is three

Here we describe the automorphism groups of associative algebras from Theorem 3.3.

Lemma 4.3. *The automorphism groups of 2-dimensional associative algebras over a field $\mathbb{F}$, $(Char(\mathbb{F}) = 3)$ are given as follows:*

1. $Aut(As^1_{13,3}) = \left\{\left(\begin{array}{cc} x & 0\\ z & 2x^2\end{array}\right) \mid x, z \in \mathbb{F}, x \neq 0\right\}$,
2. $Aut(As^2_{3,3}) = \left\{\left(\begin{array}{cc} 1 & 0\\ 0 & t\end{array}\right) \mid t \in \mathbb{F}, t \neq 0\right\}$,
3. $Aut(As^3_{3,3}) = \left\{\left(\begin{array}{cc} 1 & 0\\ 0 & t\end{array}\right) \mid t \in \mathbb{F}, t \neq 0\right\}$,
4. $Aut(As^4_{3,3}) = \left\{\left(\begin{array}{cc} 1 & 0\\ 1+2t & t\end{array}\right) \mid t \in \mathbb{F}, t \neq 0\right\}$,
5. $Aut(As^{5,1}_{3,3}(0)) = Aut\left(\left(\begin{array}{cccc} 2 & 0 & 0 & 0\\ 0 & 2 & 2 & 0\end{array}\right)\right) = \left\{I = \left(\begin{array}{cc} 1 & 0\\ 0 & 1\end{array}\right)\right\}$,
6. $Aut(As^{5,2}_{3,3}(\alpha_4)) = Aut\left(\left(\begin{array}{cccc} 2 & 0 & 0 & \alpha_4\\ 0 & 2 & 2 & 0\end{array}\right)\right) = \left\{\left(\begin{array}{cc} 1 & 0\\ z & 1\end{array}\right), z \in \mathbb{F}\right\}$ $\alpha_4 \neq 0$.

Proof. Consider $As^1_{13,3} := \begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$. Then, (11) gives

$$\left\{\begin{array}{r} y = 0,\\ x^2 - t = 0.\end{array}\right.$$

Therefore,

$$Aut(As^1_{13,3}) = \left\{\left(\begin{array}{cc} x & 0\\ z & x^2\end{array}\right) \middle| x, z \in \mathbb{F} \text{ and } x \neq 0\right\}.$$

Consider $As^2_{3,3} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. From (11), we obtain

$$\begin{cases} x-1=0, \\ y=0, \\ z=0. \end{cases}$$

Hence,

$$Aut(As^2_{3,3}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{F}, t \neq 0 \right\}.$$

Consider $As^3_{3,3} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Then,

$$\begin{cases} x=1, \\ y=0, \end{cases}$$

and

$$Aut(As^3_{3,3}) = \left\{ \begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix} \mid z, t \neq 0 \in \mathbb{F} \right\}.$$

If $A = As^4_{3,3} := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$, then (11) implies

$$\begin{cases} x=1, \\ y=0. \end{cases}$$

Therefore,

$$Aut(As^4_{3,3}) = \left\{ \begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix} \mid z, t \neq 0 \in \mathbb{F} \right\}.$$

The system of equations (11) for the group of automorphisms of $As^5_{3,3}(\alpha_4) := \begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix}$ is

$$\begin{cases} \alpha_4 z^2 + 2x^2 + x = 0, \\ \alpha_4 tz + 2xy + y = 0, \\ \alpha_4 t^2 + 2\alpha_4 x + 2y^2 = 0, \\ z(x+1) = 0, \\ (x+2)t + yz = 0, \\ \alpha_4 z + 2ty = 0. \end{cases} \quad (12)$$

The solution to the system is

$$\begin{cases} \{x=1, y=0, z=0\}, & \text{if } \alpha_4 = 0, \\ \{x=1, y=0, z=0, t=\pm 1\}, & \text{if } \alpha_4 \neq 0. \end{cases}$$

Thus,

$$Aut(As^{5,1}_{3,3}(0)) = Aut\left(\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \ t \neq 0 \in \mathbb{F} \right\},$$
$$Aut(As^{5,2}_{3,3}(\alpha_4)) = Aut\left(\begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \ z \in \mathbb{F} \right\} \ \alpha_4 \neq 0.$$

□

5 Classification of Two-dimensional Associative Dialgebras

In this section we classify all two-dimensional associative dialgebras over any basic field. As was mentioned earlier a dialgebra can be given by two 2×4 matrices;

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}, \text{ and } B = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix},$$

corresponding to the binary operations $\dashv$ and $\vdash$, respectively. The matrix equations (7) in terms of entries of A and B can be written as follows:

AXIOM 1: $A(A \otimes I) - A(I \otimes A) = 0$ is equivalent to

$$\begin{cases} \beta_1(\alpha_2 - \alpha_3) = 0, \\ \alpha_2\beta_2 - \alpha_4\beta_1 = 0, \\ (\alpha_1 - \beta_3)\alpha_2 - \alpha_3(\alpha_1 - \beta_2) = 0, \\ (\alpha_1 - \beta_2)\alpha_4 - \alpha_2(\alpha_2 - \beta_4) = 0, \\ \alpha_3\beta_3 - \alpha_4\beta_1 = 0, \\ \alpha_4(\beta_2 - \beta_3) = 0, \\ (\alpha_1 - \beta_3)\alpha_4 - \alpha_3(\alpha_3 - \beta_4) = 0, \\ \alpha_4(\alpha_2 - \alpha_3) = 0, \\ \beta_1(\beta_2 - \beta_3) = 0, \\ (\alpha_2 - \beta_4)\beta_1 - \beta_2(\alpha_1 - \beta_2) = 0, \\ (\alpha_3 - \beta_4)\beta_1 - \beta_3(\alpha_1 - \beta_3) = 0, \\ (\alpha_3 - \beta_4)\beta_2 - \beta_3(\alpha_2 - \beta_4) = 0. \end{cases} \tag{13}$$

AXIOM 2: $A(I\otimes A)-A(I\otimes B)=0$ is equivalent to

$$\begin{cases} \alpha_1^2-\alpha_1\gamma_1+\alpha_2\beta_1-\alpha_2\delta_1=0,\\ \alpha_1\alpha_2-\alpha_1\gamma_2+\alpha_2\beta_2-\alpha_2\delta_2=0,\\ \alpha_1\alpha_3-\alpha_1\gamma_3+\alpha_2\beta_3-\alpha_2\delta_3=0,\\ \alpha_1\alpha_4-\alpha_1\gamma_4+\alpha_2\beta_4-\alpha_2\delta_4=0,\\ \alpha_1\alpha_3-\alpha_3\gamma_1+\alpha_4\beta_1-\alpha_4\delta_1=0,\\ \alpha_2\alpha_3-\alpha_3\gamma_2+\alpha_4\beta_2-\alpha_4\delta_2=0,\\ \alpha_3^2-\alpha_3\gamma_3+\alpha_4\beta_3-\alpha_4\delta_3=0,\\ \alpha_3\alpha_4-\alpha_3\gamma_4+\alpha_4\beta_4-\alpha_4\delta_4=0,\\ \alpha_1\beta_1+\beta_1\beta_2-\beta_1\gamma_1-\beta_2\delta_1=0,\\ \alpha_2\beta_1-\beta_1\gamma_2+\beta_2^2-\beta_2\delta_2=0,\\ \alpha_3\beta_1-\beta_1\gamma_3+\beta_2\beta_3-\beta_2\delta_3=0,\\ \alpha_4\beta_1-\beta_1\gamma_4+\beta_2\beta_4-\beta_2\delta_4=0,\\ \alpha_1\beta_3+\beta_1\beta_4-\beta_3\gamma_1-\beta_4\delta_1=0,\\ \alpha_2\beta_3+\beta_2\beta_4-\beta_3\gamma_2-\beta_4\delta_2=0,\\ \alpha_3\beta_3+\beta_3\beta_4-\beta_3\gamma_3-\beta_4\delta_3=0,\\ \alpha_4\beta_3-\beta_3\gamma_4+\beta_4^2-\beta_4\delta_4=0. \end{cases} \tag{14}$$

AXIOM 3: $A(B\otimes I)-B(I\otimes A)=0$ is equivalent to

$$\begin{cases} \alpha_3\delta_1-\beta_1\gamma_2=0,\\ \alpha_4\delta_1-\beta_2\gamma_2=0,\\ (\delta_2-\gamma_1)\alpha_3+\gamma_2(\alpha_1-\beta_3)=0,\\ (\delta_2-\gamma_1)\alpha_4+\gamma_2(\alpha_2-\beta_4)=0,\\ \alpha_3\delta_3-\beta_1\gamma_4=0,\\ \alpha_4\delta_3-\beta_2\gamma_4=0,\\ (\gamma_3-\delta_4)\alpha_3-\gamma_4(\alpha_1-\beta_3)=0,\\ (\gamma_3-\delta_4)\alpha_4-\gamma_4(\alpha_2-\beta_4)=0,\\ (\gamma_1-\delta_2)\beta_1-\delta_1(\alpha_1-\beta_3)=0,\\ \beta_2(\gamma_1-\delta_2)-\delta_1(\alpha_2-\beta_4)=0,\\ \beta_1(\gamma_3-\delta_4)-\delta_3(\alpha_1-\beta_3)=0,\\ (\gamma_3-\delta_4)\beta_2-\delta_3(\alpha_2-\beta_4)=0. \end{cases} \tag{15}$$

AXIOM 4: $B(A \otimes I) - B(B \otimes I) = 0$ is equivalent to

$$\begin{cases} \alpha_1\gamma_1 - \gamma_1^2 + (-\delta_1 + \beta_1)\gamma_3 = 0, \\ (\beta_1 - \delta_1)\gamma_4 + \gamma_2(\alpha_1 - \gamma_1) = 0, \\ \gamma_1(\alpha_2 - \gamma_2) + \gamma_3(\beta_2 - \delta_2) = 0, \\ \alpha_2\gamma_2 - \gamma_2^2 + \gamma_4(\beta_2 - \delta_2) = 0, \\ (\beta_3 - \gamma_1 - \delta_3)\gamma_3 + \alpha_3\gamma_1 = 0, \\ \gamma_2(\alpha_3 - \gamma_3) + \gamma_4(\beta_3 - \delta_3) = 0, \\ (\alpha_4 - \gamma_4)\gamma_1 + \gamma_3(\beta_4 - \delta_4) = 0, \\ (\beta_4 - \gamma_2 - \delta_4)\gamma_4 + \alpha_4\gamma_2 = 0, \\ (\alpha_1 - \delta_3 - \gamma_1)\delta_1 + \beta_1\delta_3 = 0, \\ (\alpha_1 - \gamma_1)\delta_2 + \delta_4(-\delta_1 + \beta_1) = 0, \\ (\alpha_2 - \gamma_2)\delta_1 + \delta_3(\beta_2 - \delta_2) = 0, \\ (\alpha_2 - \gamma_2 - \delta_4)\delta_2 + \beta_2\delta_4 = 0, \\ (\alpha_3 - \gamma_3)\delta_1 + \delta_3(\beta_3 - \delta_3) = 0, \\ (\alpha_3 - \gamma_3)\delta_2 + \delta_4(\beta_3 - \delta_3) = 0, \\ (\alpha_4 - \gamma_4)\delta_1 + \delta_3(\beta_4 - \delta_4) = 0, \\ (\alpha_4 - \gamma_4)\delta_2 + \delta_4(\beta_4 - \delta_4) = 0. \end{cases} \tag{16}$$

AXIOM 5: $B(B \otimes I) - B(B \otimes I) = 0$ is equivalent to

$$\begin{cases} \delta_1(\gamma_2 - \gamma_3) = 0, \\ \gamma_2\delta_2 - \gamma_4\delta_1 = 0, \\ \gamma_1(\gamma_2 - \gamma_3) - \gamma_2\delta_3 + \gamma_3\delta_2 = 0, \\ \gamma_2^2 - \gamma_2\delta_4 - \gamma_4(\gamma_1 - \delta_2) = 0, \\ \gamma_3\delta_3 - \gamma_4\delta_1 = 0, \\ \gamma_4(\delta_2 - \delta_3) = 0, \\ \gamma_3\delta_4 - \gamma_3^2 + \gamma_4(\gamma_1 - \delta_3) = 0, \\ \gamma_4(\gamma_2 - \gamma_3) = 0, \\ \delta_1(\delta_2 - \delta_3) = 0, \\ (\gamma_2 - \delta_4)\delta_1 - \delta_2(\gamma_1 - \delta_2) = 0, \\ (\gamma_3 - \delta_4)\delta_1 - \delta_3(\gamma_1 - \delta_3) = 0, \\ (\gamma_3 - \delta_4)\delta_2 - \delta_3(\gamma_2 - \delta_4) = 0. \end{cases} \tag{17}$$

For A we take MSC of Theorems 3.1, 3.2, 3.3 for the basic field $\mathbb{F}$ with the characteristic is not 2, 3, the characteristic 2 and the characteristic 3, respectively. The entries of B we consider as unknowns:

$$\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{pmatrix}.$$

We substitute A (being associative) and B of a diassociative algebra $D := \{A, B\}$ into the systems of equations (14), (15), (16) and (17), consequentially, to get the systems of equations on the entries of MSC chosen A with unknown entries of B. Solving the systems of equations we get a diassociative algebra generated by A. Acting by the automorphism group of A we verify whether the generated by A diassociative algebras are isomorphic or not.

5.1 The characteristic of $\mathbb{F}$ is not two and three

Theorem 5.1. *Any non-trivial 2-dimensional associative dialgebra over a field $\mathbb{F}$, $(Char(\mathbb{F}) \neq 2,3)$ is isomorphic to only one of the following algebras presented their matrices of structure constants:*

1. *Diassociative algebras generated by A_{13}:*

   - $D_{13}^1 := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix}, \ \delta_1 \in \mathbb{F} \right\}.$

2. *Diassociative algebras generated by A_3:*

   - $D_3^2 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$
   - $D_3^3 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$
   - $D_3^4 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$
   - $D_3^5(\delta_1) := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \ B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix}, \ \delta_1 \in \{0,1\} \right\}.$
   - $D_3^6 := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \ B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \right\}.$
   - $D_3^7 := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \ B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \right\}.$
   - $D_3^8 := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \ B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \ \alpha_4 \in \mathbb{F} \right\}.$

Proof. Consider As_{13}^1.

AXIOM 2 gives $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$.

AXIOM 3 implies $\delta_2 = \delta_4 = 0$.

From AXIOM 4, we get $\begin{cases} \delta_1\delta_3 - \delta_3 = 0. \\ \delta_3 = 0. \end{cases}$

AXIOM 5 is satisfied automatically.

Therefore,

$$D_{13}^1 := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix} \right\},$$

is a diassociative algebra generated by As_{13}^1.

Consider $As_3^2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$$\text{AXIOM 2 implies} \begin{cases} \alpha_1 = 1, & \beta_1 = 0, & \gamma_1 = 1, \\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0, \\ \alpha_3 = 0, & \beta_3 = 0, & \gamma_3 = 0, \\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0. \end{cases}$$

$$\text{AXIOM 3 gives} \begin{cases} \alpha_1 = 1, & \beta_1 = 0, & \gamma_1 = 1, & \delta_1 = 0, \\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0, & \\ \alpha_3 = 0, & \beta_3 = 0, & \gamma_3 = 0, & \delta_3 = 0, \\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0. & \end{cases}$$

$$\text{AXIOM 4 follows} \begin{cases} \alpha_1 = 1, & \beta_1 = 0, & \gamma_1 = 1, & \delta_1 = 0, \\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0, & \\ \alpha_3 = 0, & \beta_3 = 0, & \gamma_3 = 0, & \delta_3 = 0, \\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0, & \delta_4 = 0. \end{cases}$$

Case $\delta_2 = 0$:

$$\text{From AXIOM 5 we get} \begin{cases} \alpha_1 = 1, & \beta_1 = 0, & \gamma_1 = 1, & \delta_1 = 0, \\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0, & \delta_2 = 0, \\ \alpha_3 = 0, & \beta_3 = 0, & \gamma_3 = 0, & \delta_3 = 0, \\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0, & \delta_4 = 0. \end{cases}$$

Thus, we obtain

$$D_3^2 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Case $\delta_2 = 1$:

$$\text{AXIOM 5 implies} \begin{cases} \alpha_1 = 1, & \beta_1 = 0, & \gamma_1 = 1, & \delta_1 = 0, \\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0, & \delta_2 = 1, \\ \alpha_3 = 0, & \beta_3 = 0, & \gamma_3 = 0, & \delta_3 = 0, \\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0, & \delta_4 = 0. \end{cases}$$

Therefore, we get

$$D_3^3 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

Note that the diassociative algebras D_3^2 and D_3^3 are not isomorphic since acting by the automorphism group,

$$Aut(As_3^2) = Aut\left(\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\right) = \left\{\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} \mid t \in \mathbb{F}, t \neq 0\right\},$$

to the part B of D_3^2

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} = g^{-1}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} g^{\otimes 2}, \text{ where } g = \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix},$$

we get the system of equations which is inconsistent.

Consider $As_3^3 := \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$.

$$\text{AXIOM 2 yields} \begin{cases} \alpha_1 = 1, & \beta_1 = 0, & \gamma_1 = 1, & \delta_1 = 0,\\ \alpha_2 = 0, & \beta_2 = 1, & \gamma_2 = 0, & \delta_2 = 1,\\ \alpha_3 = 0, & \beta_3 = 0, & \gamma_3 = 0, & \delta_3 = 0,\\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0, & \delta_4 = 0.\end{cases}$$

$$\text{AXIOM 3, 4, 5 produce} \begin{cases} \alpha_1 = 1, & \beta_1 = 0, & \gamma_1 = 1, & \delta_1 = 0,\\ \alpha_2 = 0, & \beta_2 = 1, & \gamma_2 = 0, & \delta_2 = 1,\\ \alpha_3 = 0, & \beta_3 = 0, & \gamma_3 = 0, & \delta_3 = 0,\\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0, & \delta_4 = 0.\end{cases}$$

Hence, we have

$$D_3^4 := \left\{A = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix},\ B = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\right\}.$$

Let A to be $As_3^4 := \begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0\end{pmatrix}$. Then,

$$\text{AXIOM 2 implies} \begin{cases} \alpha_1 = \frac{1}{2}, & \beta_1 = 0, & \gamma_1 = \frac{1}{2},\\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0,\\ \alpha_3 = 0, & \beta_3 = \frac{1}{2}, & \gamma_3 = 0,\\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0,\end{cases} \text{ and this satisfies AXIOM 3 too.}$$

AXIOM 4 gives

$$\begin{cases} \delta_1\delta_3 = 0,\\ \delta_2\delta_3 = 0,\\ 2\delta_3^2 - \delta_3 = 0,\\ \delta_4 = 0.\end{cases} \tag{18}$$

Case 1: Let $\delta_3 = 0$. Then (18) is equivalent to

$$\begin{cases} \alpha_1 = \frac{1}{2}, & \beta_1 = 0, & \gamma_1 = \frac{1}{2}, \\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0, \\ \alpha_3 = 0, & \beta_3 = \frac{1}{2}, & \gamma_3 = 0, & \delta_3 = 0, \\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0, & \delta_4 = 0. \end{cases}$$

and AXIOM 5 gives $\delta_1\delta_2 = 0$ and $\delta_2(2\delta_2 - 1) = 0$.

Case 1.1: If $\delta_2 = 0$ then we obtain

$$\begin{cases} \alpha_1 = \frac{1}{2}, & \beta_1 = 0, & \gamma_1 = \frac{1}{2}, \\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0, & \delta_2 = 0, \\ \alpha_3 = 0, & \beta_3 = \frac{1}{2}, & \gamma_3 = 0, & \delta_3 = 0, \\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0, & \delta_4 = 0, \end{cases}$$

and

$$D_3^5 := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

But if $\delta_1 \neq 0$ then these algebras are isomorphic to

$$D_3^5(\delta_1) := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix}, \delta_1 \in \{0, 1\} \right\},$$

by the base change $g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\delta_1} \end{pmatrix} \in Aut(As_3^5)$.

Case 1.2: If $\delta_2 = \frac{1}{2}$, $\delta_1 = 0$ then we get

$$\begin{cases} \alpha_1 = \frac{1}{2}, & \beta_1 = 0, & \gamma_1 = \frac{1}{2}, & \delta_1 = 0, \\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0, & \delta_2 = \frac{1}{2}, \\ \alpha_3 = 0, & \beta_3 = \frac{1}{2}, & \gamma_3 = 0, & \delta_3 = 0, \\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0, & \delta_4 = 0. \end{cases}$$

Therefore,

$$D_3^6 := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \right\}.$$

is the next associative dialgebra.

Note that the dialgebras D_3^5 and D_3^6 are not isomorphic, the fact can be seen acting by

$$Aut(As_3^4) = Aut\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix} \mid z, t \in \mathbb{F}, t \neq 0 \right\},$$

on the part B of D_3^5,

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix} = g^{-1} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} g^{\otimes 2}, \text{ where } g = \begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix}.$$

We get an inconsistent system of equations with respect to elements of g.

Case 2: $\delta_3 \neq 0$ implies $\delta_1 = 0,\ \delta_2 = 0$ and $\delta_3 = \frac{1}{2}$.

$$\text{AXIOM 4, 5 give} \begin{cases} \alpha_1 = \frac{1}{2}, & \beta_1 = 0, & \gamma_1 = \frac{1}{2}, & \delta_1 = 0, \\ \alpha_2 = 0, & \beta_2 = 0, & \gamma_2 = 0, & \delta_2 = 0, \\ \alpha_3 = 0, & \beta_3 = \frac{1}{2}, & \gamma_3 = 0, & \delta_3 = \frac{1}{2}, \\ \alpha_4 = 0, & \beta_4 = 0, & \gamma_4 = 0, & \delta_4 = 0. \end{cases}$$

As a result we obtain the following diassociative algebra,

$$D_3^7 := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},\ B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \right\}.$$

Consider $A_3^5(\alpha_4) := \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \alpha_4 \in \mathbb{F}$.

$$\text{AXIOM 2 gives} \begin{cases} \alpha_1 = \frac{1}{2}, & \beta_1 = 0, & \gamma_1 = \frac{1}{2}, & \delta_1 = 0, \\ \alpha_2 = 0, & \beta_2 = \frac{1}{2}, & \gamma_2 = 0, & \delta_2 = \frac{1}{2}, \\ \alpha_3 = 0, & \beta_3 = \frac{1}{2}, & \gamma_3 = 0, & \delta_3 = \frac{1}{2}, \\ & \beta_4 = 0, & \gamma_4 = \alpha_4, & \delta_4 = 0. \end{cases}$$

It is easy to see that this satisfies AXIOM 3, 4, 5 too.

Hence, we obtain a diassociative algebra,

$$D_3^8 := \left\{ A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \right\}.$$

□

Remark 5.1. *According to a result of* [7] *there are four classes of two-dimensional associative dialgebras over* $\mathbb{C}$ *and they are isomorphic to associative dialgebras from this paper as follows:*

$$Dias^1 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \cong D_3^5(0);$$

$$Dias^2 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\} \cong D_3^3;$$

$$Dias^3 := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}, \ \alpha \in \mathbb{C} \right\} \cong D_{13}^1;$$

$$Dias^4 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\} \cong D_3^6.$$

Since Theorem 5.1 includes the case $Char(\mathbb{C}) = 0$ *the list in* [7] *needs to be revised, accordingly.*

5.2 The characteristic of $\mathbb{F}$ is two

Theorem 5.2. *Any non-trivial* 2*-dimensional associative dialgebra over a field* $\mathbb{F}$, $(Char(\mathbb{F}) = 2)$ *is isomorphic to only one of the following algebras presented by their matrices of structure constants:*

1. $D_{12,2}^1(\delta_1) := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix}, \ \delta_1 \in \mathbb{F} \right\}.$

2. $D_{11,2}^2(\beta_1) := \left\{ A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix}, \ \beta_1 \in \mathbb{F} \right\}.$

3. $D_{6,2}^3(\beta_1) := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}, \ \beta_1 \in \{0, 1\} \right\}.$

4. $D_{6,2}^4 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$

5. $D_{6,2}^5 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$

6. $D_{4,2}^6 := \left\{ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$

7. $D_{3,2}^7 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$

8. $D_{3,2}^8 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$

9. $D_{3,2}^9 := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$

Proof. If the characteristic of the field $\mathbb{F}$ is two the associative dialgebras generated from the list of Theorem 3.2 are given as follows:

From $As^1_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, we get

$$D^1_{12,2}(\delta_1) := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix}, \ \delta_1 \in \mathbb{F} \right\}.$$

The algebra $As^2_{11,2}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1 + a^2) & 0 & 0 & 1 \end{pmatrix}$, where $a, b, \beta_1 \in \mathbb{F}$ and $b \neq 0$ produces,

$$D^2_{11,2}(\beta_1) := \left\{ A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix}, \ \beta_1 \in \mathbb{F} \right\}.$$

From $As^3_{6,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, we get

$$D^3_{6,2}(\beta_1) := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}, \ \beta_1 \in \{0, 1\} \right\},$$

$$D^4_{6,2} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\},$$

$$D^5_{6,2} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

The diassociative algebras $D^3_{6,2}$, $D^4_{6,2}$ and $D^5_{6,2}$ are not isomorphic to each others since the group of automorphisms of $As^3_{6,2}$ is trivial.

Consider $As^4_{4,2}(\beta_1) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix}$. This generates

$$D^6_{4,2} := \left\{ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

From $As^5_{3,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, we get

$$D^7_{3,2} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

and

$$D^8_{3,2} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

The algebras $D^7_{3,2}$ and $D^8_{3,2}$ are not isomorphic since there is no an element of the automorphism group,

$$Aut(As^6_{3,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix} \mid \ z, t \in \mathbb{F} \text{ and } t \neq 0 \right\},$$

sending the part B of $D^7_{3,2}$ to the part B of $D^8_{3,2}$.

Finally, from $As^6_{3,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, we get

$$D^9_{3,2} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

□

5.3 The characteristic of $\mathbb{F}$ is three

Theorem 5.3. *Any non-trivial 2-dimensional associative dialgebra over a field $\mathbb{F}$, $(Char(\mathbb{F}) = 3)$ is isomorphic to only one of the following algebras presented by their matrices of structure constants:*

1. $D^1_{13,3}(\delta_1) := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix},\ \delta_1 \in \mathbb{F} \right\}.$
2. $D^2_{3,3} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$
3. $D^3_{3,3} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$
4. $D^4_{3,3} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$
5. $D^5_{3,3}(\delta_1) := \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix},\ B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix},\ \delta_1 \in \{0, 1, 2\} \right\}.$
6. $D^6_{3,3} := \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix},\ B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \right\}.$
7. $D^7_{3,3} := \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix},\ B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \right\}.$
8. $D^8_{3,3}(\alpha_4) := \left\{ A = \begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix},\ B = \begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix},\ \alpha_4 \in \mathbb{F},\ \alpha_4 \neq 0 \right\}.$

Proof. The associative algebra $As^1_{13,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ produces,

$$D^1_{13,3}(\delta_1) := \left\{ A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix},\ \delta_1 \in \mathbb{F} \right\}.$$

From $As^2_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, we get the following diassociative algebras

$$D^2_{3,3} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$
$$D^3_{3,3} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

The algebras $D^2_{3,3}$ and $D^3_{3,3}$ are not isomorphic because there is no an element of the automorphism group,

$$Aut(As^2_{3,3}) = \left\{ \begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix} \right\},$$

sending B of $D^2_{3,3}$ to B of $D^3_{3,3}$.

The associative algebra $As^3_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ generates,

$$D^4_{3,3} := \left\{ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

From $As^4_{3,3} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$, we get

$$D^5_{3,3}(\delta_1) := \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix}, \ \delta_1 \in \{0, 1, 2\} \right\},$$

$$D^6_{3,3} := \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \right\},$$

and

$$D^7_{3,3} := \left\{ A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \right\}.$$

In order to check the isomorphisms between $D^5_{3,3}$, $D^6_{3,3}$ and $D^7_{3,3}$ we act by the elements of automorphism group,

$$Aut(As^4_{3,3}) = \left\{ \begin{pmatrix} 1 & 0 \\ 1+2t & t \end{pmatrix} \ t \neq 0 \right\},$$

to B parts of each algebras:

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix} = g^{-1} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} g^{\otimes 2}, \text{ where } g = \begin{pmatrix} 1 & 0 \\ 1+2t & t \end{pmatrix},$$
$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} = g^{-1} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} g^{\otimes 2}, \text{ where } g = \begin{pmatrix} 1 & 0 \\ 1+2t & t \end{pmatrix},$$
$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} = g^{-1} \begin{pmatrix} 2 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \end{pmatrix} g^{\otimes 2}, \text{ where } g = \begin{pmatrix} 1 & 0 \\ 1+2t & t \end{pmatrix}.$$

If write these as systems of equation with respect to element of g we get inconsistent systems of equations.

Finally, $As^5_{3,3} = \begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix}$ generates the following diassociative algebra,

$$D^8_{3,3}(\alpha_4) := \left\{ A = \begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 2 & 0 \end{pmatrix} \right\}.$$

□

6 Conclusion

In the paper first we classify all associative structures on two-dimensional vector spaces over any basic field unlike previous studies, where such a classification was done over an algebraic closed field (or over $\mathbb{C}$). The automorphism groups of the algebras found is computed. By using these we describe all associative dialgebra structures on two-dimensional vector spaces over any basic field. The list of isomorphism classes of two-dimensional associative dialgebras over $\mathbb{C}$ obtained earlier is revised.

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